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Laguerre Wavelet based Numerical Method for the Solution of Differential Equations with Variable Coeficients

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Abstract

Wavelet transforms or wavelet analysis is a recently developed mathematical tool for many problems. Wavelets also can be applied in numerical analysis. In this article, we present a Laguerre wavelet based numerical method for the solution of differential equations. The proposed technique utilizes the Laguerre wavelets basis in conjunction with collocation technique. The Laguerre wavelets basis are derived and utilized for the solution of some typical ordinary differential equations. Convergence analysis for the proposed technique has also been given. Numerical examples are provided to illustrate the efficiency and accuracy of the technique. The results show that the proposed way are quite reasonable when compare to exact solution.

Keywords:

Laguerre wavelet series, Collocation Technique, Multiresolution analysis, ODE.

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1. Introduction

Differential equations have several applications in several fields such as: physics, fluid dynamics and geophysics etc. However it is not always possible to get the solution in closed form and thus, numerical methods come into the picture.

There are several numerical methods to handle a variety of problems: Finite Difference Method, Spectral Method, Finite Element Method, Finite Volume Method and so on. Many researchers are involved in developing various numerical schemes for finding solutions of different problems [3-5,12-14,17-19,22,23,25-27,34-36].

Wavelets theory is a newly emerging area in science and engineering. It has been applied in engineering disciplines; such as signal analysis for wave form representation and segmentations, time frequency analysis, harmonic analysis etc. Wavelets permit the accurate representation of a variety of functions and operators. Moreover, wavelets establish a connection with fast numerical algorithms. Spectral methods play prominent roles in solving various kinds of differential equations. It is known that there are three most widely used spectral methods, such as tau, collocation, and Galerkin methods. Collocation methods have become increasingly popular for solving differential equations; in particular, they are very useful in providing highly accurate solutions to differential equations.

In the recent years the wavelet approach is becoming more popular in the field of numerical approximations. Different types of wavelets and approximating functions have been used for this purpose. The examples include Daubechies [11], Battle-Lemarie [38], B-spline [10], Chebyshev [1], Legendre [2, 28] and Haar wavelets [33,32,21,9], etc. On account of their simplicity, Haar wavelets have received the attention of many researchers.

A short introduction to the Haar wavelets and its applications can be found in [16,15,20,7-8, 29-31]. Laguerre wavelets, which are another type of wavelets, use Laguerre polynomials as their basis functions. They have good interpolating properties and give better accuracy for smaller number of collocation points. Applications of Laguerre wavelets for numerical approximations can be found in the references [37,24]. The basic motivation of this paper is to develop a Laguerre Wavelets Method (LWM) to solve certain differential equations. It is observed that proposed method is fully compatible with the complexity of such problems and is very user-friendly. The error estimates explicitly reveal the very high accuracy level of the suggested technique.
The rest of this paper is organized as follows. In Section 2, we discuss the properties of Laguerre wavelets. The error estimation of the Laguerre wavelets expansion is also given. In Section 3, Laguerre wavelets method of solution is given. Section 4 gives several examples to test the proposed method. A conclusion is drawn in Section 5.

2. Properties of Laguerre Wavelets

Wavelets constitute a family of functions constructed from dilation and translation of a single function called the mother wavelet. When the dilation parameter ***a*** and the translation parameter ***b*** varies continuously, families of continuous wavelets are,

$$\psi_{a,b}(x) = |a|^{\frac{-1}{2}}\, \psi\left(\frac{x-b}{a}\right), a, b \in R, a \neq 0.$$

If we restrict the parameters ***a*** and ***b*** to discrete values as $a = a_0^{-k}, b = nb_0 a_0^{-k}, a_0 > 1, b_0 > 0,$ family of discrete wavelets are,

$$\psi_{k,n}(x) = |a_0|^{\frac{1}{2}}\, \psi(a_0^k x - nb_0)$$

Where $\psi_{k,n}$ forms a wavelet basis for $L^2(R)$. In particular, when $a_0 = 2$ *and* $b_0 = 1$, then $\psi_{k,n}(x)$ forms an orthonormal basis.

Laguerre Wavelets: The Laguerre wavelets $\psi_{n,m}(x) = \psi(k,n,m,x)$ involve four arguments *n =1, 2, 3,...,*2^{k-1}, *k* is assumed any positive integer, *m* is the degree of the Laguerre polynomials and it is the normalized time. They are defined on the interval [0, 1) as,

$$\psi_{n,m}(x) = \begin{cases} 2^{\frac{k}{2}}\bar{L}_m(2^k x - 2n + 1), & \frac{n-1}{2^{k-1}} \leq x < \frac{n}{2^{k-1}} \\ 0 & otherwise \end{cases} \tag{2.1}$$

Where,

$$\bar{L}_m(x) = \frac{1}{m!} L_m(x) \tag{2.2}$$

m = 0, 1, 2... M-1. In eq. (2.2) the coefficients are used for orthonormality. Here $L_m(x)$ are the Laguerre polynomials of degree *m* with respect to the weight function w*(x) =1* on the interval [0, ∞] and satisfy the following recursive formula, $L_0(x) = 1,\ L_1(x) = 1 - x$,

$$L_{m+2}(x) = \frac{(2m+3-x)L_{m+1}(x) - (m+1)L_m(x)}{m+2},\ m = 0,1,2,3, ...,$$

$$L_2(x) = \frac{x^2}{2} - 2x + 1.$$

$$L_3(x) = \frac{x^3}{6} + 3\frac{x^2}{2} - 3x + 1.$$

$$L_4(x) = \frac{x^4}{24} - 2\frac{x^3}{3} + 3x^2 - 4x + 1.$$

$$L_5(x) = -\frac{x^5}{120} + 5\frac{x^4}{24} - 5\frac{x^3}{3} + 5x^2 - 5x + 1.$$

$$L_6(x) = \frac{x^6}{720} - \frac{x^5}{20} + 5\frac{x^4}{8} - 10\frac{x^3}{3} + \frac{15}{2}x^2 - 6x + 1.$$

$$L_7(x) = -\frac{x^7}{5040} + \frac{7}{720}x^6 - \frac{7}{40}x^5 + \frac{35}{24}x^4 - \frac{35}{6}x^3 + \frac{21}{2}x^2 - 7x + 1.$$

$$L_8(x) = \frac{x^8}{40320} - \frac{x^7}{630} + \frac{7}{180}x^6 - \frac{7}{15}x^5 + \frac{35}{12}x^4 - \frac{28}{3}x^3 + 14x^2 - 8x + 1.$$

$$L_9(x)=-\frac{x^9}{36288}+\frac{x^8}{4480}-\frac{x^7}{140}+\frac{7}{60}x^6-\frac{21}{20}x^5+\frac{21}{4}x^4-14x^3+18x^2-9x+1.$$

$$L_{10}(x)=\frac{x^{10}}{3628800}-\frac{x^9}{36288}+\frac{x^8}{896}-\frac{x^7}{42}+\frac{7}{60}x^6-\frac{21}{20}x^5+\frac{35}{4}x^4-20x^3+\frac{45}{2}x^2-10x+1.$$

$$L_{11}(x)=-\frac{x^{11}}{39916800}+\frac{11}{3628800}x^{10}-\frac{11}{72576}x^9+\frac{11}{2688}x^8-\frac{11}{168}x^7+\frac{77}{120}x^6.$$

$$-\frac{77}{20}x^5+\frac{55}{4}x^4-\frac{55}{3}x^3+\frac{55}{2}x^2-11x+1.$$

$$L_{12}(x)=\frac{x^{12}}{479001600}-\frac{x^{11}}{3326400}+\frac{11}{664800}x^{10}-\frac{11}{18144}x^9+\frac{11}{896}x^8$$

$$-\frac{11}{70}x^7+\frac{77}{60}x^6-\frac{33}{5}x^5+\frac{165}{8}x^4-\frac{110}{3}x^3+33x^2-12x+1.$$

$$L_{13}(x)=-\frac{x^{13}}{6227020800}+\frac{13}{479001600}x^{12}-\frac{13}{6652800}x^{11}+\frac{143}{1814400}x^{10}-\frac{143}{72576}x^9+\frac{143}{4480}x^8-\frac{143}{420}x^7$$

$$+\frac{143}{60}x^6-\frac{429}{40}x^5+\frac{715}{24}x^4-\frac{143}{3}x^3+39x^2-13x+1.$$

Laguerre wavelets at *k = 1, n = 1*:

$$\psi_{1,0}=\sqrt{2}.$$

$$\psi_{1,1}=2\sqrt{2}(1-x).$$

$$\psi_{1,2}=\frac{\sqrt{2}}{4}(4x^2-12x+7).$$

$$\psi_{1,3}=\frac{1}{3!}*\frac{\sqrt{2}}{3}[-4x^3+24x^2-39x+17].$$

$$\psi_{1,4}=\frac{\sqrt{2}}{4!}[\frac{2}{3}x^4-\frac{20}{3}x^3+21x^2-\frac{73}{3}x+\frac{209}{24}].$$

$$\psi_{1,5}=\frac{\sqrt{2}}{5!}[\frac{4}{15}x^5+4x^4-\frac{62}{3}x^3+\frac{136}{3}x^2-\frac{167}{4}x+\frac{773}{60}].$$

$$\psi_{1,6}=\frac{\sqrt{2}}{6!}[\frac{4}{45}x^6-\frac{28}{15}x^5+\frac{43}{3}x^4-\frac{458}{9}x^3+\frac{1045}{12}x^2-\frac{4051}{60}x+\frac{13327}{720}].$$

$$\psi_{1,7}=\frac{\sqrt{2}}{7!}[\frac{8}{315}x^7+\frac{32}{45}x^6-\frac{38}{5}x^5+\frac{358}{9}x^4-\frac{1961}{18}x^3+\frac{773}{5}x^2-\frac{37633}{360}x+\frac{65461}{2520}].$$

Function approximation: A function $y(x)$ defined over [0, 1) can be expanded as a Laguerre wavelet series as follows:

$$y(x)=\sum_{n=1}^{\infty}\sum_{m=0}^{\infty}C_{n,m}\psi_{n,m}(x) \tag{2.3}$$

where $\psi_{n,m}(x)$ is given by the equation (2.1). We approximate y(x) by truncated series,

$$y(x)\approx\sum_{n=1}^{2^{k-1}}\sum_{m=0}^{M-1}C_{n,m}\psi_{n,m}(x)=C^T\psi(x)$$

where C and $\psi(x)$ are $2^{k-1}M \times 1$ matrices given by

$$C^T = [C_{1,0},\ldots,C_{1,M-1},C_{2,0},\ldots,C_{2,M-1},\ldots,C_{2^{k-1},0},\ldots,C_{2^{k-1},M-1}].$$
$$\psi(x) = [\psi_{1,0},\ldots,\psi_{1,M-1},\psi_{2,0},\ldots,\psi_{2,M-1},\ldots,\psi_{2^{k-1},0},\ldots,\psi_{2^{k-1},M-1}].$$

Since the truncated wavelets series can be an approximate solution of differential equations one has an error function E(x) for $y(x)$ as follows:
$E(x) = |y(x) - C^T\psi(x)|$.
Convergence analysis: The following statements give the error estimation of the Laguerre wavelets expansion.

(i) If $L^2(x)$ is a vector space generated by any polynomial wavelet bases over F and $F[x]$ is Polynomial vector space over F then $F[x]$ is isomorphic to $L^2(x)$.

(ii)The series solution $y(x) = \sum_{n=1}^{\infty}\sum_{m=0}^{\infty} C_{n,m}\psi_{n,m}(x)$ defined in Eq. (2.3) using Laguerre wavelet method is converges to $y(x)$.

(iii) Laguerre wavelets $\{\Psi_{i,j}\}$ are uniformly continuous on interval I and then they are continuous.

(iv) If $\Psi_{i,j}: I \to R$ is uniformly continuous on subset I of R and $\{x_n\}$ is a Cauchy sequence in I then $\{\Psi_{i,j}(x_n)\}$ is Cauchy sequence in R. (where $\Psi_{i,j}$ is a Laguerre wavelets).

(iv) Suppose that $y(x) = C^m[0,1]$ and $C^T\psi(x)$ is the approximate solution using Laguerre wavelets. Then the error bound is,

$$||E(x)|| \le ||\frac{2}{m!\,4^m 2^{m(k-1)}} \max_{x\in[0,1]} |y^m(x)|\,||.$$

3. Laguerre Wavelets method of solution

Solution of the given differential equation can be expanded as Laguerre wavelet is as follows:

$$y(x) = \sum_{n=1}^{\infty}\sum_{m=0}^{\infty} C_{n,m}\psi_{n,m}(x)$$

Where $\psi_{n,m}(x)$ is given by the equation (2.1). We approximate y(x) by truncated series

$$y_{k,M}(x) = \sum_{n=1}^{2^{k-1}}\sum_{m=0}^{M-1} C_{n,m}\psi_{n,m}(x) = C^T\psi(x) \qquad (3.1)$$

Where, $C^T = [C_{1,0},\ldots,C_{1,M-1},C_{2,0},\ldots,C_{2,M-1},\ldots,C_{2^{k-1},0},\ldots,C_{2^{k-1},M-1}]$.

$$\psi(x) = [\psi_{1,0},\ldots,\psi_{1,M-1},\psi_{2,0},\ldots,\psi_{2,M-1},\ldots,\psi_{2^{k-1},0},\ldots,\psi_{2^{k-1},M-1}].$$

Then a total number of $2^{k-1}M$ conditions should exist to determine the $2^{k-1}M$ coefficients

$$C_{10},C_{11},\ldots,C_{1M-1},C_{20},C_{21},\ldots,C_{2M-1},\ldots,C_{2^{k-1}0},C_{2^{k-1}1},\ldots,C_{2^{k-1}M-1}.$$

Suppose, the given differential equation is of second order and it has two conditions are furnished by the initial conditions, namely

$$\begin{cases} y_{k,M}(0) = \sum_{n=1}^{2^{k-1}}\sum_{m=0}^{M-1} C_{n,m}\psi_{n,m}(0) = A \\ \frac{d}{dx}y_{k,M}(0) = \frac{d}{dx}\sum_{n=1}^{2^{k-1}}\sum_{m=0}^{M-1} C_{n,m}\psi_{n,m}(0) = B \end{cases} \qquad (3.2)$$

We see that there should be $2^{k-1}M-2$ extra conditions to recover the unknown coefficients $C_{n,m}$. These conditions can be obtained by substituting equation (3.1) in the given differential equation and using the collocation pointsx_i $(2^{k-1}M-2)$, $x_i's$ are limit points of the sequence: $\{x_i\} = \left\{\frac{1}{2}\left(1+\cos\frac{(i-1)\pi}{2^{k-1}M-1}\right)\right\}$ $i = 2,3,..$, which gives the system of equations and combining these system of equations with the eqn.(3.2) to obtain $2^{k-1}M$ system of equations from which we can compute the values for the unknown coefficients $C_{n,m}$. Same procedure is repeated for differential equations of higher order also.

4. Test Problems

Test Problem 4.1. Initially, consider the first order delay differential Equation of the form,

$$y'(x) = \frac{1}{2}e^x y(\frac{x}{2}) + \frac{1}{2}y(x),\ \ 0 \le x \le 1 \qquad (4.1)$$

with the initial condition,

$$y(0) = 1. \qquad (4.2)$$

It has the exact solution $y(x) = e^x$.

We assume,

$$y(x)=\sum_{j=0}^{M-1}c_i\psi_j \text{ for fixed } k=1,$$

$$\Rightarrow y(x)=c_1\sqrt{2}+c_2 2\sqrt{2}(1-x)+c_3\frac{\sqrt{2}}{4}[4x^2-12x+7]+c_4\frac{\sqrt{2}}{18}[-4x^3+24x^2-39x+17]$$
$$+c_5\frac{\sqrt{2}}{24}[\frac{24}{3}x^4-\frac{20}{3}x^3+21x^2-\frac{73}{3}x+\frac{209}{24}]$$
$$\Rightarrow y'(x)=-c_2 2\sqrt{2}+c_3\sqrt{2}[2x-3]+c_4\frac{\sqrt{2}}{18}[-12x^2+48x-39]+c_5\frac{\sqrt{2}}{24}[32x^3-20x^2+42x-\frac{73}{3}]$$

Substituting these values of $y(x), y'(x)$ in the given equation(4.1), We have,

$$-c_2 2\sqrt{2}+c_3\sqrt{2}[2x-3]+c_4\frac{\sqrt{2}}{18}[-12x^2+48x-39]+c_5\frac{\sqrt{2}}{24}[32x^3-20x^2+42x-\frac{73}{3}]$$
$$=\frac{1}{2}[\{e^x[c_1\sqrt{2}+c_2[-2\sqrt{2}(1-\frac{x}{2})]+c_3\frac{\sqrt{2}}{4}[4(\frac{x}{2})^2-12(\frac{x}{2})+7]+c_4\frac{\sqrt{2}}{18}[-4(\frac{x}{2})^3+24(\frac{x}{2})^2-39(\frac{x}{2})+17]$$
$$+c_5\frac{\sqrt{2}}{24}[\frac{24(\frac{x}{2})^4}{3}-\frac{20(\frac{x}{2})^3}{3}+21(\frac{x}{2})^2-\frac{73(\frac{x}{2})}{3}+\frac{209}{24}]\}$$
$$+\frac{1}{2}[c_1\sqrt{2}+c_2 2\sqrt{2}(1-x)+c_3\frac{\sqrt{2}}{4}[4x^2-12x+7]+c_4\frac{\sqrt{2}}{18}[-4x^3+24x^2-39x+17]$$
$$+c_5\frac{\sqrt{2}}{24}[\frac{24}{3}x^4-\frac{20}{3}x^3+21x^2-\frac{73}{3}x+\frac{209}{24}] \quad (4.3)$$

Since, $y(0)=1$, then we have,

$$_1\sqrt{2}+c_2 2\sqrt{2}+c_3\frac{7\sqrt{2}}{4}+c_4\frac{17\sqrt{2}}{18}+c_5\frac{209}{24}*\frac{209}{24}=1 \quad (4.4)$$

Collocating the equation (4.3) using the limit points of the sequence: $\{\frac{1}{2}(1+\frac{\cos(i1-1)}{(2^{k-1}.M-1)})\}$, Where, $i_1=2,3........$ at $k=1$ and $M=5$, then We get the following points, When,

$$i_1=2\Rightarrow x_1=0.9845$$
$$i_1=3\Rightarrow x_2=0.9388$$
$$i_1=4\Rightarrow x_3=0.8658$$
$$i_1=5\Rightarrow x_4=0.7702$$

Substituting these points in the equation (4.3), we get four algebraic systems of equations with the unknown coefficients $c_i, i=1$ to 5. Solving these five equations (4.3) and (4.4) using MATLAB, we get the value of $c_i's$ and then substituting in $y(x)=\sum_{j=0}^{M-1}c_i\psi_j$,

we get the approximate solution as,

$$y(x)=0.2603x^4+0.1499x^3+0.6807x^2+0.9927x+0.9999.$$

Test Problem 4.2. Next consider the second order Pantographic equation is of the form,

$$y''=\frac{3}{4}y(x)-y\left(\frac{x}{2}\right)-x^2+2 \text{ with } y(0)=0,\ y'(0)=0,\ 0\le x\le 1 \quad (4.5)$$

It has the exact solution:

$$y(x)=x^2. \quad (4.6)$$

We assume,

$$y(x)=\sum_{j=0}^{M-1}c_i\psi_j \text{ for a fixed } k=1.$$

$$\Rightarrow y(x)=c_1\sqrt{2}+c_2 2\sqrt{2}(1-x)+c_3\frac{\sqrt{2}}{4}[4x^2-12x+7]+c_4\frac{\sqrt{2}}{18}[-4x^3+24x^2-39x+17]$$
$$+c_5\frac{\sqrt{2}}{24}[\frac{24}{3}x^4-\frac{20}{3}x^3+21x^2-\frac{73}{3}x+\frac{209}{24}]$$
$$y'(x)=0+c_2 2\sqrt{2}(-1)+c_3\frac{\sqrt{2}}{4}[8x-12]+c_4\frac{\sqrt{2}}{18}[-12x^2+48x-39]+c_5\frac{\sqrt{2}}{24}[\frac{96}{3}x^3-\frac{60}{3}x^2+42x-\frac{73}{3}]$$
$$y''(x)=0+0+2\sqrt{2}c_3+c_4\frac{\sqrt{2}}{18}[-24x+48]+c_5\frac{\sqrt{2}}{24}[96x^2-40x+42]$$

or

$$y''(x)=2\sqrt{2}c_3+c_4\frac{4\sqrt{2}}{3}(-x+2)+c_5\frac{\sqrt{2}}{12}(48x^2-20x+21)$$

Substituting these values of $y(x), y'(x)$ and $y''(x)$ in the given equation(4.5), we have,

$$y''(x)=2\sqrt{2}c_3+c_4\frac{4\sqrt{2}}{3}(-x+2)+c_5\frac{\sqrt{2}}{12}(48x^2-20x+21)$$
$$\frac{3}{4}\{[c_1\sqrt{2}+c_2 2\sqrt{2}(1-x)+c_3\frac{\sqrt{2}}{4}[4x^2-12x+7]+c_4\frac{\sqrt{2}}{18}[-4x^3+24x^2-39x+17]$$
$$+c_5\frac{\sqrt{2}}{24}[\frac{24}{3}x^4-\frac{20}{3}x^3+21x^2-\frac{73}{3}x+\frac{209}{24}]\}$$
$$+\{c_1\sqrt{2}+c_2 2\sqrt{2}(1-\frac{x}{2})+c_3\frac{\sqrt{2}}{4}[4(\frac{x}{2})^2-12(\frac{x}{2})+7]+c_4\frac{\sqrt{2}}{18}[-4(\frac{x}{2})^3+24(\frac{x}{2})^2-39(\frac{x}{2})+17]$$
$$+c_5\frac{\sqrt{2}}{24}[\frac{24(\frac{x}{2})^4}{3}-\frac{20(\frac{x}{2})^3}{3}+21(\frac{x}{2})^2-\frac{73\frac{x}{2}}{3}+\frac{209}{24}]\}-x^2+2 \quad (4.7)$$

Since $y(0)=0$ and $y'(0)=0$ implies,

$$c_1\sqrt{2}+c_2 2\sqrt{2}+c_3\frac{7\sqrt{2}}{4}+c_4\frac{17\sqrt{2}}{18}+c_5\frac{209}{24}*\frac{209}{24}=1 \quad (4.8)$$

and $-c_2 2\sqrt{2}+c_3\sqrt{2}(2x-3)+c_4\frac{\sqrt{2}}{18}(-12x^2+48x-39)+c_5\frac{\sqrt{2}}{24}(32x^3-20x^2+42x-\frac{73}{3})=0 \quad (4.9)$

And collocating the equation (4.7) using the limit points of the following sequence: $\{\frac{1}{2}(1+\frac{\cos(i1-1)}{(2^{k-1}.M-1)})\}$,

Where, $i_1=2,3........$ at $k=1$ and $M=5$,then we get ,
$i_1=2\Rightarrow x_1=0.9845$
$i_1=3\Rightarrow x_2=0.9388$
$i_1=4\Rightarrow x_3=0.8658$

Substituting these collocating points in the equation (4.7), we get three algebraic systems of equations with the unknown coefficients $c_i, \forall\ i=1$ to 5, using MATLAB solving these systems of equations (4.7, 4.8 & 4.9), we get the value of c_1 to c_5, then substituting these in $y(x)=\sum_{j=0}^{M-1}c_i\varphi_j$.

Then we get the exact solution as $y(x)=x^2$.

Test Problem 4.3. Thirdly consider the third order pantograph equation is of the form,

$$y'''=xy''(2x)-y'(x)-y(\frac{x}{2})+x\cos(2x)+\cos(\frac{x}{2}),\quad y(0)=1, y'(0)=0, y''(0)=-1. \quad (4.10)$$

It has the exact solution $y(x)=\cos(x)$.

Using the procedure explained in section 3, we find the solution of the test problem 4.3 for different values of *M* and by increasing M values, we get more accuracy in the solution as shown in the table 1 and fig 1.

Table 1**.** Comparison of Laguerre Wavelets solution (LWM) with the exact solution of the test problem 4.3.

x	**Exact solution**	**LWM (k=1, M=5)**	**LWM (k=1, M=6)**	**LWM (k=1, M=7)**
0.1	0.995004165278026	0.995048424874032	0.995002480114178	0.995004957055140
0.2	0.980066577841242	0.980380017842200	0.980056330851874	0.980066699867935
0.3	0.955336489125606	0.956257648835982	0.955310362733543	0.955335178937694
0.4	0.921060994002885	0.922922044336683	0.921013810745427	0.921064184128300
0.5	0.877582561890373	0.880591787375435	0.877509873375584	0.877582869530998
0.6	0.825335614909678	0.829463317533201	0.825229251649921	0.825335635172135
0.7	0.764842187284489	0.769710930940766	0.764683688168228	0.764844365565878
0.8	0.696706709347165	0.701486780278748	0.696459506140211	0.696715025111881
0.9	0.621609968270664	0.624920874777590	0.621211148421527	0.621612610337926
1.0	0.540302305868140	0.540121080217562	0.540654716549813	0.540302458987522

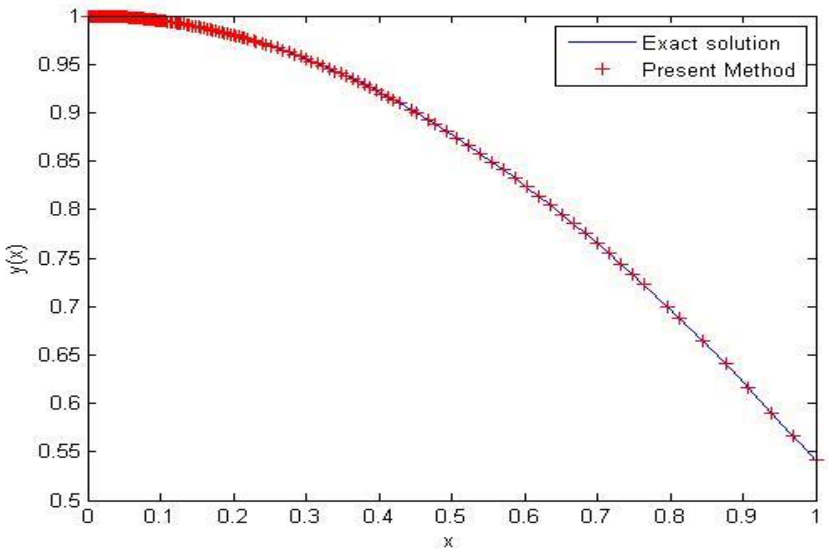

Fig.1. Comparison of Laguerre Wavelets solution (LWM, **k=1, M=7**) with the exact solution of the test problem 4.3.

Test Problem 4.4. Fourthly consider the singular initial value problem that is Lane-Emden equation is of the form,

$$y'' + \frac{2}{x}y' + y = 6 + 12x + x^2 + x^3; \quad 0 < x \leq 1, \ y(0) = 0, y'(0) = 0 \qquad (4.11)$$

It has the exact solution, $y = x^2 + x^3$**.** Solving above equation using the method presented in the section 3 for the case corresponding to k=1, M=5. After performing some manipulations, the components of the vector **C** are given by using Laguerre wavelets**:** $c_{10} = \frac{-57\sqrt{2}}{8}$, $c_{11} = \frac{-45\sqrt{2}}{16}$, $c_{12} = \frac{7\sqrt{2}}{4}$, $c_{13} = \frac{-9\sqrt{2}}{4}$, $c_{14} = 0$, and consequently we get the solution as,

$$y(x) = C^T \psi(x) = x^2 + x^3,$$

This is same as the exact solution.

Test Problem 4.5. Lastly consider the singular nonlinear Lane-Emden equation is of the form,

$$y'' + \frac{2}{x}y' + 8\,e^{y} + 4e^{\frac{y}{2}} = 0. \qquad (4.12)$$

Subjected to initial conditions are,

$$y(0) = 0, \ y'(0) = 0,$$

and its analytic solution is $y = -2\ln(1 + x^2)$. Using the procedure explained in section 3 we get the LWM solution of the test problem 4.5 and is presented in the Table 2 & Fig. 2.

Table 2. Comparison of Laguerre Wavelets solution (LWM) with the exact solution of the test problem 4.5.

X	Exact solution	Absolute Error by present method using Laguerre, Hermite

		and Legendre wavelets at k=1, M=10
0.1	-0.019900661706336	3.6640×10^{-11}
0.2	-0.078441426306563	3.4148×10^{-11}
0.3	-0.172355392482105	3.6797×10^{-11}
0.4	-0.296840010236547	3.4785×10^{-11}
0.5	-0.446287102628420	3.2490×10^{-11}
0.6	-0.614969399495921	3.3358×10^{-11}
0.7	-0.797552239914736	3.2694×10^{-11}
0.8	-0.989392483672214	2.9498×10^{-11}
0.9	-1.186653690555469	2.8965×10^{-11}
1	-0.614969399495921	2.6678×10^{-11}

Fig.2. Comparison of Laguerre Wavelets solution (LWM, **k=1, M=10**) with the exact solution of the test problem 4.5

5. Conclusions

The main goal of this paper is to develop an efficient and accurate method to solve certain differential equations those are linear or nonlinear or singular value problems. The Laguerre wavelets together with the collocation points are utilized to reduce the problem to the solution of linear or nonlinear algebraic equations. One of the main advantages of the developed algorithm is that it does not require any modification while switching from the linear case to the nonlinear case. Another one is that high accuracy approximate solutions are achieved using very small values of k and M. Illustrative examples are included to demonstrate the validity and applicability of the proposed method. According to the numerical findings are presented in the Tables and figures, we get more accurate results while increasing M. Computational work and numerical results explicitly reflect that the proposed method (LWM) is very user-friendly but extremely accurate.

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